Learning with Positive Definite Kernels: Theory, Algorithms and Applications

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Course Outline

- Introduction: Positive definite kernels and RKHS (Lecture 1)
  - Feature space vs. Function space
  - Kernel trick
  - Applications: Ridge regression, Principal component analysis
- Generalization of kernel trick to probabilities (Lecture 2)
  - Hilbert space embedding of probabilities
  - Mean element and covariance operator
  - Applications: Two-sample testing, GAN
- Approximate Kernel Methods (Lecture 3)
  - Computational vs. Statistical trade-off
  - Applications: Ridge regression, Principal component analysis

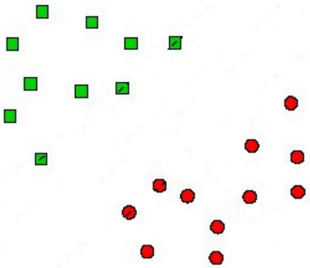
Lecture-1 Outline

- Motivating Examples
  - Nonlinear classification
  - Statistical learning
- Feature space vs. Function space
  - Kernels and properties
  - RKHS and properties
- Applications: Ridge regression, Principal component analysis
  - Kernel trick
  - Representer theorem

Motivating Example: Binary Classification

- Given: $D := \{(x_j, y_j)\}_{j=1}^{n}$, $x_j \in \mathcal{X}$, $y_j \in \{-1, +1\}$
- Goal: Learn a function $f : \mathcal{X} \to \mathbb{R}$ such that

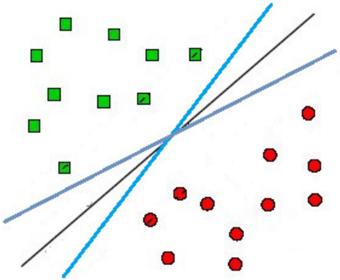
$$y_j = \mathrm{sign}(f(x_j)), \; \forall j = 1, \ldots, n.$$



Linear Classifiers

- Linear classifier: $f_{w,b}(x) = \langle w, x \rangle_2 + b$, $w, x \in \mathbb{R}^d$, $b \in \mathbb{R}$
- Find $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$y_j \left(\langle w, x_j \rangle_2 + b \right) \geq 0, \; \forall j = 1, \ldots, n.$$



- Fisher discriminant analysis, Support vector machine, Perceptron, ...

Nonlinear Classification: 1

- There is no linear classifier that separates red and blue regions.

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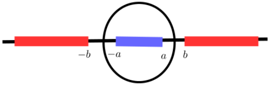

- There is no linear classifier that separates red and blue regions.
- However, the following function perfectly separates red and blue regions

$$f(x) = x^2 - r = \left\langle \underbrace{(1, -r)}_{w}, \underbrace{(x^2, 1)}_{\Phi(x)} \right\rangle_2, \ a < r < b.$$

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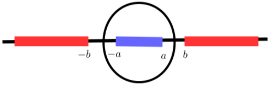

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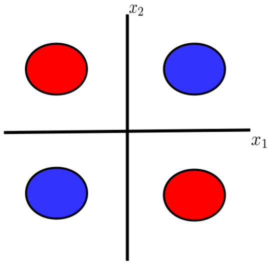
- By mapping $x \in \mathbb{R}$ to $\Phi(x) = (x^2, 1) \in \mathbb{R}^2$, the nonlinear classification problem is turned into a linear problem.
- We call Φ as the feature map (starting point of kernel trick)

Nonlinear Classification: 2



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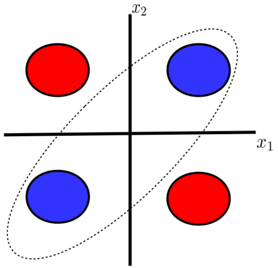

- There is no linear classifier that separates red and blue regions.
- A conic section, however, perfectly separates them

$$
\begin{aligned}
f(x) &= ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + g \\
&= \left\langle \underbrace{(a, b, c, d, e, g)}_{w}, \underbrace{(x_1^2, x_1x_2, x_2^2, x_1, x_2, 1)}_{\Phi(x)} \right\rangle_2 .
\end{aligned}
$$

- $\Phi(x) \in \mathbb{R}^6$.

Motivating Example: Statistical Learning

- Given: A set $D := \{(x_1, y_1), \ldots, (x_n, y_n)\}$ of input/output pairs drawn independently from an unknown probability distribution $\mathbf{P}$ on $X \times Y$.

- Goal: "Learn" a function $f : X \to Y$ such that $f(x)$ is a good approximation of the possible response y for an arbitrary x.

- We need a means to assess the quality of an estimated response $f(x)$ when the true input and output pair is (x, y).

- Loss function: $L : Y \times Y \to [0, \infty)$
  - Squared-loss: $L(y, f(x)) = (y - f(x))^2$
  - Hinge-loss: $L(y, f(x)) = \max(0, 1 - yf(x))$

- One common quality measure is the average loss or expected loss of f, called the risk functional i.e.,

$$\mathcal{R}_{L,\mathbf{P}}(f) := \int_{X \times Y} L(y, f(x)) \, d\mathbf{P}(x, y).$$

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Bayes Risk and Bayes Function

- Idea: Choose f that has the smallest risk.

$$f^* := \arg \inf_{f: X \to \mathbb{R}} \mathcal{R}_{L,\mathbf{P}}(f),$$

where the infimum is taken over the set of all measurable functions.

- f^* is called the Bayes function and $\mathcal{R}_{L,\mathbf{P}}(f^*)$ is called the Bayes risk.
- If $\mathbf{P}$ is known, finding f^* is often a relatively easy task and there is nothing to learn.
  - Example: $L(y, f(x)) = (y - f(x))^2$ and $L(y, f(x)) = |y - f(x)|$
  - Exercise: What is f^* for the above losses?

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Universal Consistency

- But $\mathbf{P}$ is unknown.
- However "partially known" from the training set, $D := \{(x_1, y_1), \ldots, (x_n, y_n)\}$.
- Given D, the goal is to construct $f_D : X \to \mathbb{R}$ such that

$$\mathcal{R}_{L,\mathbf{P}}(f_D) \approx \mathcal{R}_{L,\mathbf{P}}(f^*).$$

- Universally consistent learning algorithm: for all $\mathbf{P}$ on $X \times Y$, we have

$$\mathcal{R}_{L,\mathbf{P}}(f_D) \to \mathcal{R}_{L,\mathbf{P}}(f^*), \quad n \to \infty$$

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Empirical Risk Minimization

- Since $\mathbf{P}$ is unknown but is known through D, it is tempting to replace $\mathcal{R}_{L,\mathbf{P}}(f)$ by

$$\mathcal{R}_{L,D}(f) := \frac{1}{n}\sum_{i=1}^{n} L(y_i, f(x_i)),$$

called the empirical risk and find f_D by

$$f_D := \arg\min_{f:X\to\mathbb{R}} \mathcal{R}_{L,D}(f)$$

- Is it a good idea?
- No! Choose f_D such that $f_D(x) = y_i,\ x = x_i,\ \forall\, i$ and $f_D(x) = 0,$ *otherwise*.
- $\mathcal{R}_{L,D}(f_D) = 0$ but can be very far from $\mathcal{R}_{L,\mathbf{P}}(f^*)$.

Overfitting!!

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Overfitting!!

Method of Sieves (Structural Risk Minimization)

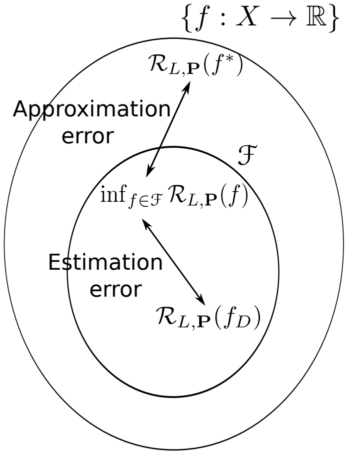
- How to avoid overfitting: Perform ERM on a small set $\mathcal{F}$ of functions $f : X \to Y$ (class of smooth functions) where the size of $\mathcal{F}$ grows appropriately with n.
- Do minimization over $\mathcal{F}$:

$$f_D := \arg \inf_{f \in \mathcal{F}} \mathcal{R}_{L,D}(f)$$

- Total error: Define $\mathcal{R}^*_{L,\mathbf{P},\mathcal{F}} := \inf_{f \in \mathcal{F}} \mathcal{R}_{L,\mathbf{P}}(f)$

$$\mathcal{R}_{L,\mathbf{P}}(f_D) - \mathcal{R}^*_{L,\mathbf{P}} = \overbrace{\mathcal{R}_{L,\mathbf{P}}(f_D) - \mathcal{R}^*_{L,\mathbf{P},\mathcal{F}}}^{\text{Estimation error}} + \overbrace{\mathcal{R}^*_{L,\mathbf{P},\mathcal{F}} - \mathcal{R}^*_{L,\mathbf{P}}}^{\text{Approximation error}}$$

Approximation and Estimation Errors



How to choose $\mathcal{F}$?

$$f_D = \arg\inf_{f\in\mathcal{F}} \mathcal{R}_{L,D}(f) = \arg\inf_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n} L(y_i, \underbrace{f(x_i)}_{\delta_{x_i}(f)})$$

- An <u>evaluation functional</u> is a linear functional δ_x that evaluates each function in the space at the point x, i.e.,

$$\delta_x(f) = f(x),\ \forall f \in \mathcal{F}.$$

- Bounded evaluation functional: An evaluation functional is bounded if there exists a M such that

$$|\delta_x(f)| = |f(x)| \leq M_x \|f\|_{\mathcal{F}},\ \forall x, \in \mathcal{X},\ f \in \mathcal{F}$$

  where $\mathcal{F}$ is a normed vector space (continuity of δ_x).

- Evaluation functionals are not always bounded.
- Example: $L^2[a,b]$
  - $\|f\|_2$ remains the same if f is changed at a countable set of points.

Choice of $\mathcal{F}$

- Various choices for $\mathcal{F}$ (with evaluation functional bounded):
  - Lipschitz functions
  - Bounded Lipschitz functions
  - Bounded continuous functions
- If $\mathcal{F}$ is a Hilbert space of functions with bounded evaluation functionals for all $x \in \mathcal{X}$, computationally efficient estimators can be obtained.

Reproducing Kernel Hilbert Space

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Reproducing Kernel Hilbert Space

Summary

Points of view:

- Feature map, Φ: trick to achieve non-linear methods from linear ones
- Function space, $\mathcal{F}$: statistical generalization and computational efficiency

History

- Mathematics (Functional analysis): Introduced in 1907 by Stanisław Zaremba for studying boundary value problems; developed by Mercer, Szegö, Bergman, Bochner, Moore, Aronszajn; reached maturity by late 1950's.
- Statistics: Started by Emmanuel Parzen (early 1960's) and pursued by Wahba (between 1970 and 1990).
- Pattern recognition/Machine learning: Started by Aizerman, Braverman and Rozonoer (1964) but fury of activity following the work of Boser, Guyon and Vapnik (1992).

Other areas: Signal processing, control, probability theory, stochastic processes, numerical analysis

Kernels

(Feature space view point)

Hilbert Space

Inner product: Let $\mathcal{H}$ be a vector space over $\mathbb{R}$. A map $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is an inner product on $\mathcal{H}$ if

- Linear in the first argument: for any $f_1, f_2, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$

$$\langle \alpha f_1 + \beta f_2, g \rangle_{\mathcal{H}} = \alpha \langle f_1, g \rangle_{\mathcal{H}} + \beta \langle f_2, g \rangle_{\mathcal{H}};$$

- Symmetric: for any $f, g \in \mathcal{H}$,

$$\langle f, g \rangle_{\mathcal{H}} = \langle g, f \rangle_{\mathcal{H}};$$

- Positive definiteness: for any $f \in \mathcal{H}$,

$$\langle f, f \rangle_{\mathcal{H}} \geq 0 \quad \text{and} \quad \langle f, f \rangle_{\mathcal{H}} = 0 \Leftrightarrow f = 0.$$

Define $\| \cdot \|_{\mathcal{H}} := \langle \cdot, \cdot \rangle_{\mathcal{H}}$ as the norm on $\mathcal{H}$ induced by the inner product.

A complete (by adding the limits of all Cauchy sequences w.r.t. $\| \cdot \|_{\mathcal{H}}$) inner product space is defined as a Hilbert space.

Measure of similarity

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Kernel

(Steinwart and Christmann, 2008)

Throughout, we assume that $\mathcal{X}$ is a non-empty set (input space)

Kernel: A function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is called a kernel if there exists a Hilbert space $\mathcal{H}$ and a map $\Phi : \mathcal{X} \to \mathcal{H}$ such that

$$k(x, x') := \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}}, \quad \forall x, x' \in \mathcal{H}.$$

Φ: Feature map and $\mathcal{H}$: Feature space

Non-uniqueness of Φ and $\mathcal{H}$: Suppose $k(x, x') = xx'$, $x, x' \in \mathbb{R}$. Then

$$\Phi_1(x) = x \quad \text{and} \quad \Phi_2(x) = \frac{1}{2}(x, x)$$

are feature maps with corresponding feature spaces being $\mathbb{R}$ and $\mathbb{R}^2$.

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Properties

- For any $\alpha > 0$, αk is a kernel.

$$\alpha k(x, x') = \alpha \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}} = \langle \sqrt{\alpha}\Phi(x), \sqrt{\alpha}\Phi(x') \rangle_{\mathcal{H}}.$$

- Conic sum of kernels is a kernel: If $(k_i)_{i=1}^m$ is a collection of kernels, then for any $(\alpha_i)_{i=1}^m \subset \mathbb{R}^+$, $\sum_{i=1}^m \alpha_i k_i$ is a kernel.

$$\begin{aligned}\sum_{i=1}^m \alpha_i k_i(x, x') = \sum_{i=1}^m \alpha_i \langle \Phi_i(x), \Phi_i(x') \rangle_{\mathcal{H}_i} &= \sum_{i=1}^m \langle \sqrt{\alpha_i}\Phi_i(x), \sqrt{\alpha_i}\Phi_i(x') \rangle_{\mathcal{H}_i} \\ &= \langle \tilde{\Phi}(x), \tilde{\Phi}(x') \rangle_{\tilde{\mathcal{H}}}\end{aligned}$$

for all $x, x' \in \mathcal{X}$ where

$$\tilde{\Phi}(x) = (\sqrt{\alpha_1}\Phi_1(x), \ldots, \sqrt{\alpha_m}\Phi_m(x)) \quad \text{and} \quad \tilde{\mathcal{H}} = \underbrace{\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m}_{\text{direct sum}}.$$

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Properties

- Difference of kernels is NOT a kernel:
  - Suppose $\exists\, x \in \mathcal{X}$ such that $k_1(x,x) - k_2(x,x) < 0$.
  - If $k_1 - k_2$ is a kernel, then $\exists\, \Phi$ and $\mathcal{H}$ such that for all $x, x' \in \mathcal{H}$,

$$k_1(x,x') - k_2(x,x') = \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}}.$$

  - Choose $x = x'$.

- Product of kernels is a kernel: If k_1 and k_2 are kernels, then $k_1 \cdot k_2$ is a kernel.

$$\begin{aligned} k((x_1,x_2),(x_1',x_2')) &= k_1(x_1,x_1') \cdot k_2(x_2,x_2') \\ &= \langle \Phi_1(x_1), \Phi_1(x_1') \rangle_{\mathcal{H}_1} \cdot \langle \Phi_2(x_2), \Phi_2(x_2') \rangle_{\mathcal{H}_2} \\ &= \langle \Phi_1(x_1) \otimes \Phi_2(x_2), \Phi_1(x_1') \otimes \Phi_2(x_2') \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} \end{aligned}$$

where $\otimes$ denotes the tensor product.

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$$\begin{aligned} k((x_1,x_2),(x_1',x_2')) &= k_1(x_1,x_1') \cdot k_2(x_2,x_2') \\ &= \langle \Phi_1(x_1), \Phi_1(x_1') \rangle_{\mathcal{H}_1} \cdot \langle \Phi_2(x_2), \Phi_2(x_2') \rangle_{\mathcal{H}_2} \\ &= \langle \Phi_1(x_1) \otimes \Phi_2(x_2), \Phi_1(x_1') \otimes \Phi_2(x_2') \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} \end{aligned}$$

where $\otimes$ denotes the tensor product.

Properties

- Suppose k_1 is defined on $\{0, 1\}$ and k_2 is defined on $\{A, B, C\}$. Then clearly $k_1 \cdot k_2$ is defined on $\{0, 1\} \times \{A, B, C\}$.
- Suppose for simplicity, we assume $\mathcal{H}_1 = \mathbb{R}^2$ and $\mathcal{H}_2 = \mathbb{R}^5$. Then

$$\begin{aligned} k_1(x_1, x_1') \cdot k_2(x_2, x_2') &= \langle \Phi_1(x_1), \Phi_1(x_1') \rangle_{\mathbb{R}^2} \cdot \langle \Phi_2(x_2), \Phi_2(x_2') \rangle_{\mathbb{R}^5} \\ &= \Phi_1^\top(x_1')\Phi_1(x_1)\Phi_2^\top(x_2)\Phi_2(x_2') \\ &= \mathrm{Tr}\left(\underbrace{\Phi_2(x_2')\Phi_1^\top(x_1')}_{\mathbb{R}^2 \to \mathbb{R}^5} \underbrace{\Phi_1(x_1)\Phi_2^\top(x_2)}_{\mathbb{R}^5 \to \mathbb{R}^2} \right) \\ &= \left\langle \Phi_1(x_1)\Phi_2^\top(x_2), \Phi_1(x_1')\Phi_2^\top(x_2') \right\rangle_{\mathbb{R}^2 \otimes \mathbb{R}^5} \\ &=: \langle \Phi_1(x_1) \otimes \Phi_2(x_2), \Phi_1(x_1') \otimes \Phi_2(x_2') \rangle_{\mathbb{R}^2 \otimes \mathbb{R}^5} \end{aligned}$$

where $\mathbb{R}^2 \otimes \mathbb{R}^5$ is the space of 2×5 matrices.

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Properties

- For any arbitrary function $f : \mathcal{X} \to \mathbb{R}$,

$$\tilde{k}(x, x') = f(x)k(x, x')f(x') \tag{1}$$

is a kernel.

$$\begin{aligned}\tilde{k}(x, x') = f(x)k(x, x')f(x') &= f(x)\langle \Phi(x), \Phi(x')\rangle_{\mathcal{H}} f(x') \\ &= \langle \underbrace{f(x)\Phi(x)}_{\Phi_f(x)}, \underbrace{f(x')\Phi(x')}_{\Phi_f(x')} \rangle_{\mathcal{H}}.\end{aligned}$$

- $k(x, x) \geq 0$: $k(x, x) = \langle \Phi(x), \Phi(x)\rangle_{\mathcal{H}} = \|\Phi(x)\|^2_{\mathcal{H}} \geq 0.$
- Cauchy-Schwartz: $|k(x, y)| \leq \sqrt{k(x, x)}\sqrt{k(x', x')}$

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Properties

- Infinite dimensional feature map:

$$k(x, x') = \sum_{i \in I} \phi_i(x)\phi_i(x') \quad \text{is a kernel}$$

  if $\|(\phi_i(x))_i\|^2_{\ell_2(I)} := \sum_{i \in I} \phi_i^2(x) < \infty$ for all $x \in \mathcal{X}$.

- Proof:

$$k(x, x') = \langle \Phi(x), \Phi(x') \rangle_{\mathcal{H}}$$

  where $\Phi(x) = (\phi_i(x))_{i \in I}$ and $\mathcal{H} = \ell_2(I)$, which is the space of square summable sequences on I.

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Examples

- Polynomial kernel: $k(x,x') = (c + \langle x, x' \rangle_2)^m$, $x, x' \in \mathbb{R}^d$ for $c \geq 0$ and $m \in \mathbb{N}$. Use binomial theorem to expand, apply sum and product rules.
- Linear kernel: $c = 0$ and $m = 1$.
- Exponential kernel: $k(x,x') = \exp(\langle x, x' \rangle_2)$, $x, x' \in \mathbb{R}^d$.

  Use Taylor series expansion,

$$k(x,x') = \exp(\langle x, x' \rangle_2) = \sum_{i=0}^{\infty} \frac{\langle x, x' \rangle_2^i}{i!}.$$

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Positive Definiteness

- Kernels are symmetric and positive definite: EASY
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- Symmetric and positive definite functions are kernels: NOT OBVIOUS

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Positive Definiteness: Translation Invariant Kernels

Let $\mathcal{X} = \mathbb{R}^d$. A kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^d$ is said to be translation invariant if

$$k(x, y) = \psi(x - y), \; x, y \in \mathbb{R}^d,$$

where ψ is a positive definite function on $\mathbb{R}^d$.

- Bochner's theorem provides a complete characterization for the positive definiteness of ψ.
- A continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$ is positive definite if and only if ψ is the Fourier transform of a finite non-negative Borel measure Λ, i.e.,

$$\psi(x) = \underbrace{\int_{\mathbb{R}^d} e^{\sqrt{-1}\langle x,\omega\rangle_2} \, d\Lambda(\omega)}_{\text{Characteristic function of } \Lambda}.$$

Given a continuous integrable function ψ, i.e., $\int_{\mathbb{R}^d} |\psi(x)| \, dx < \infty$, compute

$$\hat{\psi}(\omega) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\sqrt{-1}\langle \omega,x\rangle_2} \psi(x) \, dx.$$

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Exercise

- Show that
$$\psi(x) = (1 - |x|)1_{[-1,1]}(x),\ x \in \mathbb{R}$$
is positive definite.

- Show that
$$\psi(x) = \frac{1}{2}(2 - |x|)^2 1_{\{(2-|x|)\in[0,1]\}} + \left(1 - \frac{x^2}{2}\right) 1_{[-1,1]}(x),\ x \in \mathbb{R}$$
is NOT positive definite.

So far...

Kernels $\Leftrightarrow$ Symmetric and positive definite functions

Reproducing Kernel Hilbert Space

(Function space view point)

Reproducing Kernel Hilbert Space

- A Hilbert space $\mathcal{H}$ of <u>real-valued functions</u> on $\mathcal{X}$ is said to be a reproducing kernel Hilbert space (RKHS) with $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ as the reproducing kernel, if
  - $\forall x \in \mathcal{X},\ k(\cdot, x) \in \mathcal{H}$;
  - $\forall x \in \mathcal{X}, \forall f \in \mathcal{H},\ \langle f, k(\cdot, x)\rangle_{\mathcal{H}} = f(x)$.
- The reproducing kernel (r.k.) k of $\mathcal{H}$ is a kernel:

$$k(x, x') = \left\langle \underbrace{k(\cdot, x)}_{\Phi(x)}, \underbrace{k(\cdot, x')}_{\Phi(x')} \right\rangle_{\mathcal{H}}, \quad x, x' \in \mathcal{X}.$$

  We refer to $\Phi(x) = k(\cdot, x)$ as the canonical feature map.
- Every r.k. is a symmetric and positive definite function.
- The evaluation functional is bounded:

$$\begin{aligned}|\delta_x(f)| = |f(x)| = |\langle f, k(\cdot, x)\rangle_{\mathcal{H}}| &\le \|k(\cdot, x)\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \\ &= \sqrt{k(x, x)}\|f\|_{\mathcal{H}}, \ \forall x \in \mathcal{X},\ f \in \mathcal{H}.\end{aligned}$$

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Reproducing Kernel Hilbert Space

- Every Hilbert function space with a reproducing kernel is an RKHS.
- The converse is true: Every RKHS has a unique reproducing kernel.
- (Moore-Aronszajn Theorem)

  If k is a positive definite kernel, then there exists a unique RKHS with k as the reproducing kernel.

  (Proof: Define $H = \{f : f = \sum_{i=1}^{n} \alpha_i k(\cdot, x_i),\ \alpha_i \in \mathbb{R},\ x_i \in \mathcal{X}\}$ endowed with the bilinear form

$$\langle f, g \rangle_H = \sum_{i,j=1}^{n} \alpha_i \beta_j k(x_i, x_j).$$

  Verify that $\langle \cdot, \cdot \rangle_H$ is an inner product and $\langle f, k(\cdot, x) \rangle_H = f(x)$ for any $f \in H$. Complete H to obtain an RKHS.)

Kernels $\Leftrightarrow$ Positive definite & symmetric functions $\Leftrightarrow$ RKHS

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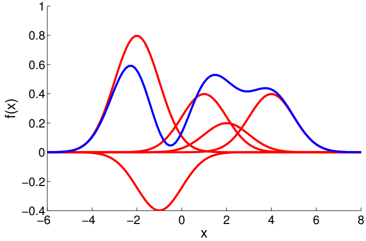
Functions in the RKHS

- $\mathcal{H} = \overline{\text{span}\{k(\cdot, x) \;:\; x \in \mathcal{X}\}}$ (linear span of kernel functions)
  - Example: $f(x) = \sum_{i=1}^{m} \alpha_i k(x, x_i)$ for arbitrary $m \in \mathbb{N}$, $\{\alpha_i\} \subset \mathbb{R}$, $x \in \mathcal{X}$ and $\{x_i\} \subset \mathcal{X}$.

$$k(x, y) = e^{-\|x-y\|^2/2\sigma^2}$$



Picture credit: A. Gretton

Properties of RKHS

- k is bounded if and only every $f \in \mathcal{H}$ is bounded.
- If $\int_{\mathcal{X}} \sqrt{k(x,x)}\, d\mu(x) < \infty$, then for every $f \in \mathcal{H}$, $\int_{\mathcal{X}} f(x)\, d\mu(x) < \infty$.
- Every $f \in \mathcal{H}$ is continuous if and only if $k(\cdot, x)$ is continuous for all $x \in \mathcal{X}$.
- Every $f \in \mathcal{H}$ is m-times continuously differentiable if k is m-times continuously differentiable.

k controls the properties of $\mathcal{H}$

Explicit Realization of RKHS

- $\mathcal{X} = \mathbb{R}^d$ and $k(x, y) = \psi(x - y)$ where ψ is a positive definite function.
- Assume ψ satisfies $\int_{\mathbb{R}^d} |\psi(x)|\, dx < \infty$. Denote $\hat{\psi}$ to be the Fourier transform of ψ.
- Define $L^2(\mathbb{R}^d) := \{f : \int_{\mathbb{R}^d} |f(x)|^2\, dx < \infty\}$. Then

$$\mathcal{H} = \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d} \frac{|\hat{f}(\omega)|^2}{\hat{\psi}(\omega)}\, d\omega < \infty \right\}$$

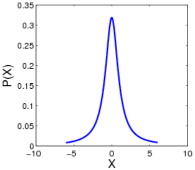
endowed with

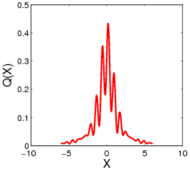
$$\langle f, g \rangle_{\mathcal{H}} = (2\pi)^{-d/2} \int \frac{\hat{f}(\omega)\overline{\hat{g}(\omega)}}{\hat{\psi}(\omega)}\, d\omega$$

is an <u>RKHS with k as the r.k.</u>

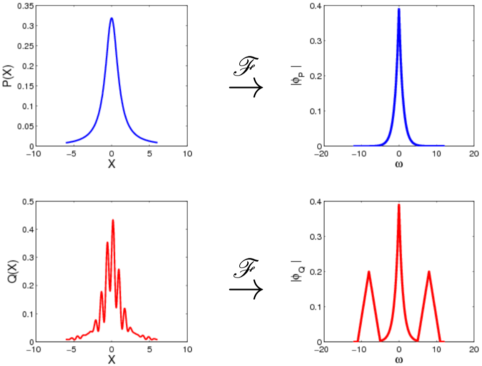
(Wendland, 2005)

Fourier Transform

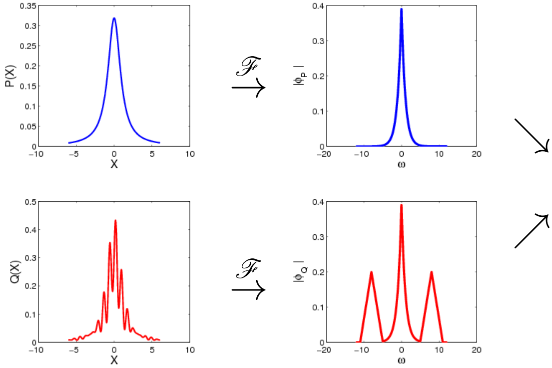




Fourier Transform



Fourier Transform



Smooth function

$\Updownarrow$

Fast rate of decay of
Fourier transform

Gaussian RKHS

- Gaussian kernel:

$$k(x,y) = \psi(x-y) = e^{-\|x-y\|_2^2/\gamma^2},\ x,y \in \mathbb{R}^d$$

- Fourier transform:

$$\hat{\psi}(\omega) = \left(\frac{\gamma^2}{2}\right)^{d/2} e^{-\frac{\gamma^2\|\omega\|_2^2}{4}},\ \omega \in \mathbb{R}^d$$

-

$$\mathcal{H}_\gamma(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) \ :\ \underbrace{\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 e^{\frac{\gamma^2\|\omega\|_2^2}{4}}\, d\omega}_{\|f\|^2_{\mathcal{H}_\gamma}} < \infty \right\}$$

Fast decay of $\hat{\psi} \Rightarrow$ Smooth $\mathcal{H}$

Gaussian RKHS

- Gaussian kernel:

$$k(x, y) = \psi(x - y) = e^{-\|x-y\|_2^2/\gamma^2},\ x, y \in \mathbb{R}^d$$

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- $\{f : \|f\|_{\mathcal{H}_\gamma} \leq \alpha\} \subset \{f : \|f\|_{\mathcal{H}_\gamma} \leq \beta\} \subset \mathcal{H}_\gamma$ for any $\alpha < \beta$.

More smoothness

Sobolev RKHS

- Laplacian kernel:

$$k(x,y) = \psi(x-y) = \sqrt{\frac{\pi}{2}}e^{-|x-y|},\ x,y \in \mathbb{R}$$

- Fourier transform:

$$\hat{\psi}(\omega) = \frac{1}{1+|\omega|^2},\ \omega \in \mathbb{R}$$

-

$$\mathcal{H}_1^2(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) \ : \ \underbrace{\int_{\mathbb{R}} |\hat{f}(\omega)|^2(1+|\omega|^2)\,d\omega}_{\|f\|^2_{\mathcal{H}_1^2}} < \infty \right\}$$

- $\{f : \|f\|_{\mathcal{H}_1^2} \le \alpha\} \subset \{f : \|f\|_{\mathcal{H}_1^2} \le \beta\} \subset \mathcal{H}_1^2$ for any $\alpha < \beta$.

Extension to $\mathbb{R}^d$: Matérn Kernel

Summing Up

- Kernels: Feature map Φ and feature space $\mathcal{H}$
- Positive definiteness and Bochner's theorem
- RKHS: Canonical feature map $\Phi(x) = k(\cdot, x)$
- Kernels $\Leftrightarrow$ Positive definite & symmetric functions $\Leftrightarrow$ RKHS
- Properties of k control the properties of the RKHS.
- Smoothness

Application: Ridge Regression

(Kernel Trick: Feature map point of view)

Ridge regression

- Given: $\{(x_i, y_i)\}_{i=1}^n$ where $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$
- Task: Find a linear regressor $f = \langle w, \cdot \rangle_2$ s.t. $f(x_i) \approx y_i$,

$$\min_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^n (\langle w, x_i \rangle_2 - y_i)^2 + \lambda \|w\|_2^2 \quad (\lambda > 0)$$

- Solution: For $\mathbf{X} := (x_1, \ldots, x_n) \in \mathbb{R}^{d \times n}$ and $\mathbf{y} := (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$,

$$w = \underbrace{\frac{1}{n} \left(\frac{1}{n} \mathbf{X}\mathbf{X}^\top + \lambda I_d \right)^{-1} \mathbf{X}\mathbf{y}}_{primal}$$

- Easy:

$$\left(\frac{1}{n} \mathbf{X}\mathbf{X}^\top + \lambda I_d \right) \mathbf{X} = \mathbf{X} \left(\frac{1}{n} \mathbf{X}^\top \mathbf{X} + \lambda I_n \right)$$

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Ridge regression

- Prediction: Given $t \in \mathbb{R}^d$

$$f(t) = \langle w, t \rangle_2 = \mathbf{y}^\top \mathbf{X}^\top \left(\mathbf{X}\mathbf{X}^\top + n\lambda I_d \right)^{-1} t$$
$$= \mathbf{y}^\top \left(\mathbf{X}^\top \mathbf{X} + n\lambda I_n \right)^{-1} \mathbf{X}^\top t$$

- How does $\mathbf{X}^\top\mathbf{X}$ look like?

$$\mathbf{X}^\top\mathbf{X} = \underbrace{\begin{bmatrix} \langle x_1, x_1 \rangle_2 & \langle x_1, x_2 \rangle_2 & \cdots & \langle x_1, x_n \rangle_2 \\ \langle x_2, x_1 \rangle_1 & \langle x_2, x_2 \rangle_2 & \cdots & \langle x_2, x_n \rangle_2 \\ \vdots & \langle x_i, x_j \rangle_2 & \ddots & \vdots \\ \langle x_n, x_1 \rangle_1 & \langle x_n, x_2 \rangle_2 & \cdots & \langle x_n, x_n \rangle_2 \end{bmatrix}}_{\text{Matrix of inner products: Gram Matrix}}$$

Ridge regression

- Prediction: Given $t \in \mathbb{R}^d$

$$
\begin{aligned}
f(t) = \langle w, t \rangle_2 &= \mathbf{y}^\top \mathbf{X}^\top \left(\mathbf{X}\mathbf{X}^\top + n\lambda I_d \right)^{-1} t \\
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\end{aligned}
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Kernel Ridge regression: Feature Map and Kernel Trick

- Given: $\{(x_i, y_i)\}_{i=1}^n$ where $x_i \in \mathcal{X}$, $y_i \in \mathbb{R}$
- Task: Find a regressor $f \in \mathcal{H}$ (some feature space) s.t. $f(x_i) \approx y_i$.
- Idea: Map x_i to $\Phi(x_i)$ and do linear regression,

$$\min_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} (\langle f, \Phi(x_i) \rangle_{\mathcal{H}} - y_i)^2 + \lambda \|f\|_{\mathcal{H}}^2 \quad (\lambda > 0)$$

- Solution: For $\Phi(\mathbf{X}) := (\Phi(x_1), \ldots, \Phi(x_n)) \in \mathbb{R}^{\dim(\mathcal{H}) \times n}$ and $\mathbf{y} := (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$,

$$f = \underbrace{\frac{1}{n}\left(\frac{1}{n}\Phi(\mathbf{X})\Phi(\mathbf{X})^\top + \lambda I_{\dim(\mathcal{H})}\right)^{-1}\Phi(\mathbf{X})\mathbf{y}}_{primal}$$

$$= \underbrace{\frac{1}{n}\Phi(\mathbf{X})\left(\frac{1}{n}\Phi(\mathbf{X})^\top\Phi(\mathbf{X}) + \lambda I_n\right)^{-1}\mathbf{y}}_{dual}$$

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Kernel Ridge regression: Feature Map and Kernel Trick

- Prediction: Given $t \in \mathcal{X}$

$$f(t) = \langle f, \Phi(t) \rangle_{\mathcal{H}} = \frac{1}{n}\mathbf{y}^\top \Phi(\mathbf{X})^\top \left(\frac{1}{n}\Phi(\mathbf{X})\Phi(\mathbf{X})^\top + \lambda I_{\dim(\mathcal{H})} \right)^{-1} \Phi(t)$$
$$= \frac{1}{n}\mathbf{y}^\top \left(\frac{1}{n}\Phi(\mathbf{X})^\top \Phi(\mathbf{X}) + \lambda I_n \right)^{-1} \Phi(\mathbf{X})^\top \Phi(t)$$

As before

$$\Phi(\mathbf{X})^\top \Phi(\mathbf{X}) = \underbrace{\begin{bmatrix} \langle \Phi(x_1), \Phi(x_1) \rangle_{\mathcal{H}} & \cdots & \langle \Phi(x_1), \Phi(x_n) \rangle_{\mathcal{H}} \\ \langle \Phi(x_2), \Phi(x_1) \rangle_{\mathcal{H}} & \cdots & \langle \Phi(x_2), \Phi(x_n) \rangle_{\mathcal{H}} \\ \vdots & \ddots & \vdots \\ \langle \Phi(x_n), \Phi(x_1) \rangle_{\mathcal{H}} & \cdots & \langle \Phi(x_n), \Phi(x_n) \rangle_{\mathcal{H}} \end{bmatrix}}_{k(x_i, x_j) = \langle \Phi(x_i), \Phi(x_j) \rangle_{\mathcal{H}}}$$

and

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Feature Map and Kernel Trick: Remarks

- The primal formulation requires the knowledge of feature map Φ (and of course $\mathcal{H}$) and these could be infinite dimensional.
- Suppose we have access to a kernel function, k (Recall: not easy to verify that k is a kernel). Then the dual formulation is entirely determined by k (Gram matrix or kernel matrix).
- Linear regression in the dual uses a linear kernel.

Kernel trick or heuristic

Replace $\langle x_i, x_j \rangle_2$ in your linear method by $k(x_i, x_j)$ where k is your favorite kernel

Application: Principal Component Analysis

(Kernel Trick: Feature map point of view)

Principal Component Analysis

- Dimensionality reduction
- Given: $\{(x_i)\}_{i=1}^{n}$ where $x_i \in \mathbb{R}^d$
- Task: Find a low-dimensional representation for (x_i).

$$
\begin{aligned}
&\max_{\|w\|_2=1} \mathrm{Var}\left(\langle w, x_1\rangle_2, \langle w, x_2\rangle_2, \ldots, \langle w, x_n\rangle_2\right)\\
&\equiv \max_{\|w\|_1=1} \frac{1}{n}\sum_{i=1}^{n}\langle w, x_i\rangle_2^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\langle w, x_i\rangle_2\right)^2\\
&\equiv \max_{\|w\|_1=1} w^\top \hat{\boldsymbol{\Sigma}} w
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\boldsymbol{\Sigma}} &:= \frac{1}{n}\sum_{i=1}^{n} x_i x_i^\top - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^\top\\
&= \frac{1}{n}\mathbf{X}\left(I_d - \frac{1}{n}\mathbf{1}\mathbf{1}^\top\right)\mathbf{X}^\top =: \mathbf{X}\mathbf{H}\mathbf{X}^\top.
\end{aligned}
$$

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$$
\begin{aligned}
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&= \frac{1}{n}\mathbf{X}\left(I_d - \frac{1}{n}\mathbf{1}\mathbf{1}^\top\right)\mathbf{X}^\top =: \mathbf{X}\mathbf{H}\mathbf{X}^\top.
\end{aligned}
$$

Principal Component Analysis

- Solution: Find the eigenvector corresponding to the maximum eigenvalue of $\hat{\Sigma}$,

$$\begin{aligned} \hat{\boldsymbol{\Sigma}} w &= \lambda_1 w \\ \mathbf{X}\mathbf{H}\mathbf{X}^\top w &= \lambda_1 w \\ \underbrace{\mathbf{X}^\top \mathbf{X}}_{\mathbf{K}} \mathbf{H} \underbrace{\mathbf{X}^\top w}_{\alpha} &= \lambda_1 \mathbf{X}^\top w \\ \mathbf{K}\mathbf{H}\alpha &= \lambda_1 \alpha \end{aligned}$$

- The eigenvalues of $\mathbf{KH}$ are the same as that of $\hat{\boldsymbol{\Sigma}}$
- $$\alpha = \mathbf{X}^\top w \Longrightarrow \mathbf{X}\mathbf{H}\alpha = \mathbf{X}\mathbf{H}\mathbf{X}^\top w = \hat{\boldsymbol{\Sigma}} w = \lambda_1 w$$

  The eigenvector of $\hat{\boldsymbol{\Sigma}}$ can be computed from the eigenvector of $\mathbf{KH}$ as

  $$w = \frac{1}{\lambda_1}\mathbf{X}\mathbf{H}\alpha$$

- Kernel PCA (Schölkopf et al., 1998):

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Principal Component Analysis

- Solution: Find the eigenvector corresponding to the maximum eigenvalue of $\hat{\Sigma}$,

$$\hat{\mathbf{\Sigma}} w = \lambda_1 w$$

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$$\mathbf{KH}\alpha = \lambda_1 \alpha$$

- The eigenvalues of $\mathbf{KH}$ are the same as that of $\hat{\mathbf{\Sigma}}$
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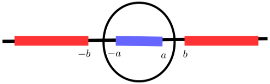
Feature Map and Kernel Trick

Same idea yields: (Schölkopf and Smola, 2002)

- Linear SVM $\rightarrow$ Kernel SVM
- Fisher discriminant analysis (FDA) $\rightarrow$ Kernel FDA
- Canonical correlation analysis (CCA) $\rightarrow$ Kernel CCA

many more ...

Revisiting Nonlinear Classification: 1

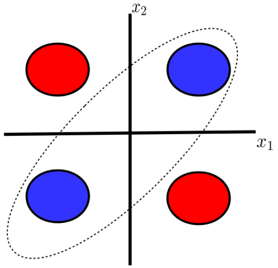


- The following function perfectly separates red and blue regions

$$f(x) = x^2 - r = \left\langle \underbrace{(1, -r)}_{w}, \underbrace{(x^2, 1)}_{\Phi(x)} \right\rangle_2 , \; a < r < b.$$

- Apply kernel trick with $k(x, y) = x^2 y^2 + 1$.

Revisiting Nonlinear Classification: 2



- A conic section, however, perfectly separates them

$$f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + g$$
$$= \left\langle \underbrace{(a, b, c, d, e, g)}_{w}, \underbrace{(x_1^2, x_1x_2, x_2^2, x_1, x_2, 1)}_{\Phi(x)} \right\rangle_2 .$$

- Apply kernel trick with $k(x, y)$. Exercise: Find the kernel $k(x, y)$.

Application: Ridge Regression

(Representer Theorem: Function space point of view)

Learning Theory: Revisit

- Empirical risk: $\mathcal{R}_{L,D}(f) := \frac{1}{n}\sum_{i=1}^{n} L(y_i, f(x_i))$

$$f_D := \arg\min_{f:X\to\mathbb{R}} \mathcal{R}_{L,D}(f)$$

- To avoid overfitting: Perform ERM on a small set $\mathfrak{F}$ of functions (class of smooth functions)

$$f_D := \arg\inf_{f\in\mathfrak{F}} \mathcal{R}_{L,D}(f)$$

- Choice of $\mathcal{F}$: Evaluation functionals are bounded.

$$|\delta_x(f)| = |f(x)| \le M_x \|f\|_{\mathfrak{F}},\ \forall\, x \in \mathcal{X},\ f \in \mathfrak{F}$$

Pick $\mathcal{F} = \{f : \|f\|_{\mathcal{H}} \le \alpha\}$; $\mathcal{H}$ is an RKHS

Classification with Lipschitz functions (von Luxburg and Bousquet, JMLR 2004)

Penalized Estimation

- We have

$$f_D = \arg \inf_{\|f\|_{\mathcal{H}} \leq \alpha} R_{L,D}(f)$$

$$= \arg \inf_{\|f\|_{\mathcal{H}} \leq \alpha} \frac{1}{n} \sum_{i=1}^{n} L(y_i, f(x_i))$$

- In the Lagrangian formulation, we have

$$f_D = \arg \inf_{f \in \mathcal{H}} R_{L,D}(f) + \lambda \|f\|_{\mathcal{H}}^2$$

$$= \arg \inf_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} L(y_i, f(x_i)) + \lambda \|f\|_{\mathcal{H}}^2$$

where $\lambda > 0$.

Optimization over (possibly infinite dimensional) function space

Representer Theorem

Consider the penalized estimation problem,

$$\inf_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} L(y_i, f(x_i)) + \lambda \theta(\|f\|_{\mathcal{H}})$$

where $\theta : [0, \infty) \to \mathbb{R}$ is a non-decreasing function.

- (Kimeldorf and Wahba, 1971; Schölkopf et al., ALT 2001) The solution to the above minimization problem is achieved by a function of the form

$$f = \sum_{i=1}^{n} \alpha_i k(\cdot, x_i),$$

where $(\alpha_i)_{i=1}^{n} \subset \mathbb{R}$.

The infinite dimensional optimization problem reduces to a finite dimensional optimization problem in $\mathbb{R}^n$.

Proof

- Decomposition:
$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp},$$
where $\mathcal{H}_0 = \mathrm{span}\{k(\cdot, x_1), \ldots, k(\cdot, x_n)\}$, $\mathcal{H}_0^{\perp}$: orthogonal complement. Decompose
$$f = f_0 + f^{\perp}$$
accordingly.

- The loss function L does not change by replacing f with f_0 because
$$f(x_i) = \langle f, k(\cdot, x_i)\rangle_{\mathcal{H}} = \langle f_0, k(\cdot, x_i)\rangle_{\mathcal{H}} + \underbrace{\langle f^{\perp}, k(\cdot, x_i)\rangle_{\mathcal{H}}}_{=0}.$$

- Penalty term:
$$\|f_0\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \qquad \Rightarrow \qquad \theta(\|f_0\|_{\mathcal{H}}) \leq \theta(\|f\|_{\mathcal{H}}).$$

- Thus the optimum lies in $\mathcal{H}_0$.

Kernel Ridge Regression

- $f : \mathcal{X} \to \mathbb{R}$ and $L(y, f(x)) = (y - f(x))^2$ (Squared loss)

$$\inf_{f \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \langle f, k(\cdot, x_i)\rangle_{\mathcal{H}}\right)^2 + \lambda \|f\|_{\mathcal{H}}^2$$

- By representer theorem, the solution is of the form $f = \sum_{i=1}^{n} \alpha_i k(\cdot, x_i)$ which on substitution yields

$$\inf_{\alpha} \frac{1}{n} \|\mathbf{Y} - \mathbf{K}\alpha\|^2 + \lambda \alpha^\top \mathbf{K} \alpha$$

where $\mathbf{K}$ is the Gram matrix with $\mathbf{K}_{ij} = k(x_i, x_j)$.

- Solution: $\hat{\alpha} = (\mathbf{K} + n\lambda I_n)^{-1}\mathbf{Y}$ (assuming $\mathbf{K}$ is invertible). For any $t \in \mathcal{X}$,

$$\hat{f}(t) = \sum_{i=1}^{n} \hat{\alpha}_i k(t, x_i) = \mathbf{Y}^\top (\mathbf{K} + n\lambda I_n)^{-1} \mathbf{k}_t,$$

where $(\mathbf{k}_t)_i := k(t, x_i)$. (Same solution as the feature map view point)

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Application: Principal Component Analysis

(Representer Theorem: Function space point of view)

Principal Component Analysis

- Given: $\{(x_i)\}_{i=1}^n$ where $x_i \in \mathcal{X}$.
- $$\begin{aligned}
&\min_{\|f\|_{\mathcal{H}}=1} \operatorname{Var}\left(f(x_1),\ldots,f(x_n)\right) \\
&= \min_{\|f\|_{\mathcal{H}}=1} \operatorname{Var}\left(\langle f, k(\cdot,x_1)\rangle_{\mathcal{H}},\ldots,\langle f, k(\cdot,x_n)\rangle_{\mathcal{H}}\right) \\
&= \min_{\|f\|_{\mathcal{H}}=1} \frac{1}{n}\sum_{i=1}^n \langle f, k(\cdot,x_i)\rangle_{\mathcal{H}}^2 - \left\langle f, \frac{1}{n}\sum_{i=1}^n k(\cdot,x_i)\right\rangle_{\mathcal{H}}^2 \\
&= \min_{\|f\|_{\mathcal{H}}=1} \langle f, \hat{\Sigma} f\rangle_{\mathcal{H}}
\end{aligned}$$

  where

  $$\hat{\Sigma} := \frac{1}{n}\sum_{i=1}^n k(\cdot,x_i)\otimes k(\cdot,x_i) - \left(\frac{1}{n}\sum_{i=1}^n k(\cdot,x_i)\right)\otimes\left(\frac{1}{n}\sum_{i=1}^n k(\cdot,x_i)\right)$$

  is the covariance operator.
- The solution to the above problem is the eigenfunction of $\hat{\Sigma}$ corresponding to the maximum eigenvalue. (CAUTION!!)

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How to choose $\mathcal{H}$?

Large RKHS: Universal Kernel/RKHS

- Universal kernel (Steinwart, JMLR 2001): A kernel k on a compact metric space, $\mathcal{X}$ is said to be universal if the RKHS, $\mathcal{H}$ is dense (w.r.t. uniform norm) in the space of continuous functions on $\mathcal{X}$.

  Any continous function on $\mathcal{X}$ can be approximated arbitrarily by a function in $\mathcal{H}$.

- (Steinwart and Christmann, 2008) For certain conditions on L, if k is universal, then

$$\inf_{f \in \mathcal{H}} \mathcal{R}_{L,\mathbf{P}}(f) = \mathcal{R}_{L,\mathbf{P}}(f^*),$$

  i.e., approximation error is zero.

  - Squared loss, Hinge loss,...

Large RKHS: Universal Kernel/RKHS

- Universal kernel (Steinwart, JMLR 2001): A kernel k on a compact metric space, $\mathcal{X}$ is said to be universal if the RKHS, $\mathcal{H}$ is dense (w.r.t. uniform norm) in the space of continuous functions on $\mathcal{X}$.

  Any continous function on $\mathcal{X}$ can be approximated arbitrarily by a function in $\mathcal{H}$.

- (Steinwart and Christmann, 2008) For certain conditions on L, if k is universal, then

$$\inf_{f \in \mathcal{H}} \mathcal{R}_{L,\mathbf{P}}(f) = \mathcal{R}_{L,\mathbf{P}}(f^*),$$

  i.e., approximation error is zero.

  - Squared loss, Hinge loss,...

When is k Universal?

k is universal if and only if

$$\int_{\mathcal{X}} \int_{\mathcal{X}} k(x, y)\, d\mu(x)\, d\mu(y) > 0$$

for all non-zero finite signed measures, μ on $\mathcal{X}$.

(Carmeli et al., 2010; S et al., 2011)

Generalization of strictly positive definite kernels

- In Lecture 2, we will explore more by relating it to the Hilbert space embedding of measures.
- Examples: Gaussian, Laplacian, etc. (No finite dimensional RKHS is universal!!)

Advanced Topics

- Consistency and convergence rates of KRR (Caponnetto and De Vito, 2007; Steinwart et al., 2009)
- Consistency and convergence rates of KPCA (Blanchard et al., 2007; Rudi et al., 2013)
- Spectral regularization
- Stochastic gradient descent methods for KRR

Questions

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Suggested Readings

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